Formalizing Actuarial Mathematics in Proof Assistants

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What Is Actuarial Mathematics?

Actuarial mathematics is a branch of applied mathematics used to assess the financial risks associated with undesirable events. It is related to

1. calculus,
2. probability theory,
3. statistics,
4. financial theory.

The traditional actuarial roles are considered as

1. determining the prices of insurance products,
2. estimating the liabilities of a company associated with insurance contracts.

Recently, the risk management skills of actuaries are increasingly valued across a wider range of businesses.

Pricing a Term Life Insurance

"Term life insurance offers a death benefit to beneficiaries if the insured passes away within a set period, known as the term." [2]

Assumption

1. amount insured: \$10000
2. entry age: 30 years old
3. policy period: 1 year
4. annual mortality rate: 1%
5. annual interest rate: 2%

The expected payment after 1 year is $\$10000 \times 0.01 = \$100$. If the insurance company earns a 2% investment yield annually, the required amount for this insurance can be discounted:

$$\frac{\$100}{1+0.02} \approx \$98.$$

We call $1/(1+0.02)$ the discount factor.

Definition

The expected present value of a term life insurance on a person aged x payable at the end of the year of death within n years is denoted by $A^1_{x:\overline{n}|}$ per unit insurance amount:

$$A^1_{x:\overline{n}|} := \sum_{k=1}^{n} v^k \cdot {}_{k-1|}q_x,$$

where

1. v denotes the discount factor, and
2. ${}_{k-1|}q_x$ denotes the probability that a person aged x dies in the k-th year.

In the example above, $A^1_{30:\overline{1}|} \approx 0.0098$. This can be thought of as a single premium (the money you pay to the insurance company) of this term life insurance.

Definition

The expected present value of a life annuity due (i.e., the annual payments of 1 unit at the beginning of each year while the person is alive) on a person aged x within n years is denoted by $\ddot{a}_{x:\overline{n}|}$:

$$\ddot{a}_{x:\overline{n}|} := \sum_{k=1}^{n} v^{k-1} \cdot {}_{k-1}p_x,$$

where

1. v denotes the discount factor, and
2. ${}_tp_x$ denotes the probability that a person aged x survives for t years.

If you pay the premium annually, the level premium $P^1_{x:\overline{n}|}$ should satisfy the equivalence principle:

$$P^1_{x:\overline{n}|} \cdot \ddot{a}_{x:\overline{n}|} = A^1_{x:\overline{n}|} \quad \text{(premiums = benefits).}$$

International Actuarial Notation

Various actuarial symbols have been used worldwide since at least the 20th century [3].

a_x = an annuity, first payment at the end of a year, to continue during the life of (x).

$\ddot{a}_x = 1 + a_x$ = an 'annuity-due' to continue during the life of (x), the first payment to be made at once.

A_x = an assurance payable at the end of the year of death of (x).

Note. $e_x = a_x$ at rate of interest $i = 0$.

A letter or number at the lower left corner of the principal symbol denotes the number of years involved in the probability or benefit in question. Thus:

${}_np_x$ = the probability that (x) will live n years.

${}_nq_x$ = the probability that (x) will die within n years.

Note. When $n = 1$ it is customary to omit it, as shown on page 2, provided no ambiguity is introduced.

${}_nE_x = v^n {}_np_x$ = the value of an endowment on (x) payable at the end of n years if (x) be then alive.

If the letter or number comes before a perpendicular bar it shows that a period of deferment is meant. Thus:

${}_{n|}q_x$ = the probability that (x) will die in a year, deferred n years; that is, that he will die in the $(n+1)$th year.

${}_{n|}a_x$ = an annuity on (x) deferred n years; that is, that the first payment is to be made at the end of $(n+1)$ years.

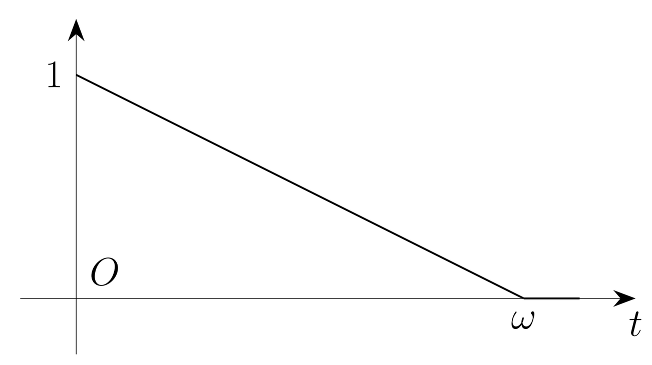
${}_{n|t}a_x$ = an intercepted or deferred temporary annuity on (x) deferred n years and, after that, to run for t years.

Previous Study: Rocq Formalization

The first attempt to formalize actuarial mathematics (life insurance mathematics) was in Rocq (Coq):

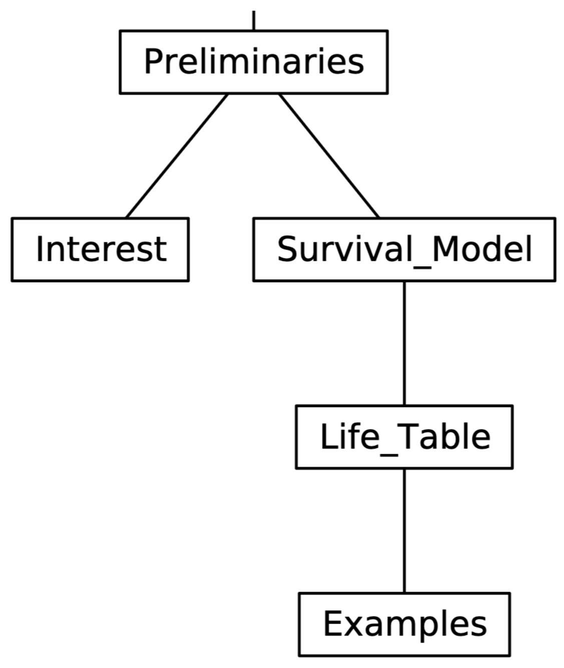
`https://github.com/Yosuke-Ito-345/Actuary`

I used Coquelicot (an analysis library), which resulted in the C^1-smoothness limitation in the survival probability $t \mapsto \Pr(X > t)$ of the insured, where X is the life span. This is inconvenient because it excludes elementary functions such as:



Isabelle Formalization

Isabelle has rich analysis libraries including Lebesgue integration and probability theory. The recent stable release is available as "Actuarial Mathematics" in the Archive of Formal Proofs (AFP) [1].



In formalizing actuarial mathematics, I had to prepare many lemmas including formulas on the Lebesgue-Stieltjes integration. This year, I uploaded the new entry "Lebesgue-Stieltjes Integral" to AFP. `https://www.isa-afp.org/entries/Lebesgue_Stieltjes_Integral.html` This entry provides basic lemmas on calculating the Lebesgue-Stieltjes integral, e.g.,

$$\int g(x)\, dF(x) = \int g(x) F'(x)\, dx.$$

In the current development, I am working on formalizing premium calculations. To this end, I need to formalize the present values of various insurance products, e.g., the continuous term life annuity:

$$\bar{a}_{x:\overline{n}|} = \int_0^n v^t \cdot {}_tp_x \, dt.$$

```
proposition
  a'_term_life_set_integrable: "set_integrable lborel {0..n} (λt. $v.^t * $p_{t&x})" and
  a'_term_life_calc: "$a'_{x;n⌉} = (LBINT t:{0..n}. $v.^t * $p_{t&x})"
  if "n ≥ 0" "x < $ψ" for n x :: real
  using that a'_defer_term_life_set_integrable[of 0] a'_defer_term_life_calc by simp+
```

Developing MathComp-Analysis

MathComp-Analysis is an actively maintained and developed analysis library for Rocq.

`https://github.com/math-comp/analysis`

It already includes Lebesgue integration and the basic parts of probability theory. I am partially involved in its development, e.g.,

1. introduced the cumulative distribution function $\mathrm{cdf}_X(r) := \Pr(X \le r)$,
2. introduced the complementary cumulative distribution function $\mathrm{ccdf}_X(r) := \Pr(X > r)$,
3. proved the tail expectation formula:

$$\mathrm{E}[X] = \int_0^\infty \mathrm{ccdf}_X(r)\, dr - \int_{-\infty}^0 \mathrm{cdf}_X(r)\, dr.$$

```
Lemma expectation_cdf_ccdf (X : {RV P >-> R}) :
  Lfun P 1 X ->
  'E_P[X] = \int[mu]_(r in `[0%R, +oo[) ccdf X r
          - \int[mu]_(r in `]-oo, 0%R[) cdf X r.
```

In the current development, I am trying to formalize the change-of-variables formula for the Lebesgue integral.

Lemma

For any function $f : [a,b] \to \mathbb{R}$ of class C^1 and any bounded Borel measurable function $g : \mathbb{R} \to \mathbb{R}$, we have

$$\int_{f(a)}^{f(b)} g(x)\, dx = \int_a^b g(f(t)) f'(t)\, dt.$$

Here both sides are oriented Lebesgue integrals, i.e., the sign is taken into account.

I may need to prepare more lemmas, but I also plan to formalize actuarial mathematics in Rocq.

Industrial Applications

Error detection

In the future, we could apply proof assistants and formalized actuarial mathematics in industry practices. For example, if we can formally verify the solutions to actuarial examinations in advance, the questions can be guaranteed to be error-free.

Example (Society of Actuaries, 2022)

You are given the following survival function for a newborn:

$$S_0(x) = (1 - 0.01x)^{0.5}, \quad 0 \le x \le 100.$$

Calculate $1000\mu_{25}$.

This solution can be formalized as:

```
lemma SoA_LTAM_2022_Spring_MCQ_No3:
  assumes "⋀x::real. 0 ≤ x ⟹ x ≤ 100 ⟹ ccdf (distr 𝔐 borel X) x = (1 - 0.01*x).^0.5"
  shows "¦1000*$μ_25 - 6.7¦ < 0.05"
```

Software Verification

Actuarial modeling software is used to evaluate the risks of insurance products, etc.

1. Fully verifying such software would involve great difficulties, particularly in formally specifying its behavior.
2. The formalized actuarial mathematics could help us create the specification document and verify the source code.

How AI Helps Us

Auto-formalization

By *Auto-formalization*, I mean automatically generating formal proofs (maybe from natural language proofs). Aristotle by Harmonic is highly capable of solving mathematical problems with formal proofs in Lean.

`https://aristotle.harmonic.fun/`

Aristotle formally proved the lemma presented below in less than an hour, whereas it took me more than a week. Such AI systems could accelerate formalizing actuarial mathematics.

Lemma

Let $v \in \mathbb{R}$ satisfy $0 \le v < 1$, and define a function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) := \lceil x \rceil \cdot v^x$. Prove that

$$\lim_{m \to \infty} \sum_{k=0}^{\infty} \frac{1}{m} \cdot f\left(\frac{k+1}{m}\right) = \int_0^\infty f(x)\, dx.$$

Formal Semantic Translation

Implicit parameters are often omitted in actuarial notation, e.g.,

$$\bar{a}_{x:\overline{n}|} = \int_0^n v^t \cdot {}_tp_x \, dt$$

depends not only on the age x and the period n but also on the interest rate i and the life span X. These parameters must eventually be specified before we can verify actuarial documents with proof assistants. AI may be well suited to such translations from real-world texts into a formal language.

References

[1] Yosuke Ito. Actuarial mathematics. *Archive of Formal Proofs*, 2024. `https://isa-afp.org/entries/Actuarial_Mathematics.html`, Formal proof development.

[2] Julia Kagan. A guide to term life insurance: Types, advantages, and disadvantages. `https://www.investopedia.com/terms/t/termlife.asp`, 2025.

[3] Francis S. Perryman. International actuarial notation. In *Proceedings of the Casualty Actuarial Society*, volume 36, pages 123–131, 1949. Also available as `https://www.casact.org/sites/default/files/database/proceed_proceed49_49123.pdf`.

Disclaimer

1. The contents presented here are solely the speaker's opinions and do not reflect the views of any affiliations.
2. There are some inaccuracies in the explanation of actuarial mathematics, as priority is given to intuitive understanding.
3. Future versions of the formalized libraries may differ from what is presented here.
4. I used a generative AI (Claude) to refine my English text and to get some advice on the layout, but I take the responsibility for the content of this poster.